



Impulsive Resonance Periodic Problems of First Order

J. J. NIETO

Departamento de Análisis Matemático, Facultad de Matemáticas
Universidad de Santiago de Compostela
15782 Santiago de Compostela, Spain

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Abstract—We prove a new existence theorem for a nonlinear periodic boundary value problem for a first-order differential equation with impulses at fixed moments. It includes the cases when the nonlinearity and the impulsive functions are either bounded or have sublinear growth. © 2002 Elsevier Science Ltd. All rights reserved.

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1. INTRODUCTION

Let $T > 0$, $J = [0, T]$, $0 = t_0 < t_1 < \dots < t_p < t_{p+1} = T$, $J' = J - \{t_1, \dots, t_p\}$, and consider the following nonlinear impulsive periodic boundary value problem:

$$\begin{aligned} u'(t) + F(t, u(t)) &= 0, & \text{a.e. } t \in J', \\ u(t_j^+) &= u(t_j^-) + I_j(u(t_j^-)), & j = 1, \dots, p, \\ u(0) &= u(T), \end{aligned} \quad (1)$$

where $F : J \times \mathbb{R} \rightarrow \mathbb{R}$ is an impulsive Carathéodory function (see below), and the impulses $I_j : \mathbb{R} \rightarrow \mathbb{R}$, $j = 1, \dots, p$, are continuous.

Let $u : J \rightarrow \mathbb{R}$. For $j = 1, \dots, p$, define the functions $u_j : (t_j, t_{j+1}) \rightarrow \mathbb{R}$, $u_j(t) = u(t)$. Consider the following Banach spaces:

$$E = \{u : J \rightarrow \mathbb{R} : u_j \in L^1(t_j, t_{j+1}), j = 0, 1, \dots, p\}$$

and

$$\begin{aligned} \mathcal{E} &= \{u : J \rightarrow \mathbb{R} : u_j \in W^{1,1}(t_j, t_{j+1}), j = 0, 1, \dots, p, \\ &\text{there exist the limits } u(0^+) = u(0), u(t_j^-) = u(t_j), u(t_j^+) \} \end{aligned}$$

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with the norms

$$\|u\|_E = \sum_{j=0}^p \|u_j\|_{L^1(t_j, t_{j+1})}, \quad \|u\|_E = \sum_{j=0}^p \|u_j\|_{W^{1,1}(t_j, t_{j+1})}.$$

A function F is an impulsive Carathéodory function if

- $F(\cdot, u)$ is measurable for every $u \in \mathbf{R}$,
- $F(t, \cdot)$ is continuous for a.e. $t \in J'$,
- for every $r > 0$ there exists $h_r \in L^1(J)$ such that

$$|F(t, u)| \leq h_r(t), \quad \text{for a.e. } t \in J' \text{ and every } u \in \mathbf{R}, \quad |u| \geq r,$$

- and for every $(t, u) \in J' \times \mathbf{R}$, $j = 1, \dots, p$, the following limits exist:

$$\lim_{t \rightarrow t_j^-} F(t, u) = F(t_j, u), \quad \lim_{t \rightarrow t_j^+} F(t, u).$$

By a solution of (1), we mean a function $u \in \mathcal{E}$ satisfying (1).

For a general theory on impulsive differential equations, see the monographs [1,2] and [3, Chapter 15]. Some present-day trends are discussed, for instance, in [4,5]. Interesting and recent aspects of impulsive systems are considered, among many papers, in [6–8].

In [9], we study problem (1) in the the nonresonance case, i.e.,

$$F(t, u) = \lambda u - f(t, u), \quad (2)$$

with $\lambda \neq 0$, $f : J \times \mathbf{R} \rightarrow \mathbf{R}$ continuous on $J' \times \mathbf{R}$. We proved existence results when the nonlinear function f and the impulses I_j , $j = 1, \dots, p$ are bounded or they have at most sublinear growth.

The paper is organized as follows. In Section 2, we recall some general results for a linear equation with periodic boundary value conditions and given impulses at the points t_j , $j = 1, \dots, p$. Then, in Section 3, we prove our main existence theorem for the nonlinear problem (1) and we show how it improves previous results.

2. LINEAR PROBLEM

For $\alpha, \sigma \in L^1(J)$, consider the following linear periodic boundary value problem:

$$u'(t) + \alpha(t)u(t) = \sigma(t), \quad \text{a.e. } t \in J; \quad u(0) = u(T). \quad (3)$$

This problem has a unique solution if and only if $\int_0^T \alpha(s) ds \neq 0$. In such a case, the solution of (3) is given by

$$u(t) = \int_0^T g_\alpha(t, s) \sigma(s) ds, \quad (4)$$

where g_α is the Green's function

$$g_\alpha(t, s) = \begin{cases} \frac{e^{-[a(t)-a(s)]}}{1 - e^{-a(T)}}, & 0 \leq s \leq t \leq T, \\ \frac{e^{-[a(T)+a(t)-a(s)]}}{1 - e^{-a(T)}}, & 0 \leq t < s \leq T, \end{cases}$$

with

$$a(t) = \int_0^t \alpha(s) ds, \quad t \in J.$$

Note that if $a(T) > 0$, then $g_\alpha(t, s) > 0$ for every $t, s \in J$. If, in addition, $\beta \in L^1(J)$, $\int_0^T \beta(s) ds > 0$, and $\alpha \geq \beta$ a.e. on J , then $g_\alpha \leq g_\beta$. We now consider problem (3) with some given impulses $\theta_j \in \mathbf{R}$, $j = 1, \dots, p$,

$$\begin{aligned} u'(t) + \alpha(t)u(t) &= \sigma(t), & \text{a.e. } t \in J'; & \quad u(0) = u(T); \\ u(t_j^+) &= u(t_j^-) + \theta_j, & j &= 1, \dots, p. \end{aligned} \quad (5)$$

Then, the solution of (5) is given by (see [9, Lemma 2.1])

$$u(t) = \int_0^T g_\alpha(t, s)\sigma(s) ds + \sum_{j=1}^p g_\alpha(t, t_j)\theta_j.$$

3. NONLINEAR PROBLEM

We now consider the nonlinear problem (1) in its equivalent form (for $k \neq 0$)

$$\begin{aligned} u'(t) + ku(t) &= ku(t) - F(t, u(t)), & \text{a.e. } t \in J', \\ u(t_j^+) &= u(t_j^-) + I_j(u(t_j^-)), & j = 1, \dots, p, \\ u(0) &= u(T). \end{aligned} \quad (6)$$

Thus, $u \in \mathcal{E}$ is a solution of (1) if and only if

$$u = B_k u,$$

where

$$[B_k u](t) = \int_0^T g_k(t, s)[ku(s) - F(s, u(s))] ds + \sum_{j=1}^p g_k(t, t_j)I_j(u(t_j^-)). \quad (7)$$

Hence, for $\sigma \in L^1(J)$, we have that $B_k \sigma$ is the solution of the following impulsive problem:

$$\begin{aligned} u'(t) + ku(t) &= k\sigma(t) - F(t, \sigma(t)), & \text{a.e. } t \in J', \\ u(t_j^+) &= u(t_j^-) + I_j(u(t_j^-)), & j = 1, \dots, p, \\ u(0) &= u(T). \end{aligned} \quad (8)$$

As in Lemma 3.2 of [9], the operator $B_k : E \rightarrow E$ is compact.

THEOREM 1. *Consider the nonlinear problem (1) and suppose that there exists $r > 0$ and $k > 0$ such that*

$$\frac{F(t, u)}{u} \geq k > 0, \quad \text{a.e. } t \in J, \quad \text{and for every } |u| \geq r, \quad (9)$$

and for every $j = 1, \dots, p$,

$$\lim_{|u| \rightarrow +\infty} \frac{I_j(u)}{u} = 0. \quad (10)$$

Then, there exists at least one solution of (1).

PROOF. We shall employ a well-known technique of nonlinear analysis [10] and show that the solutions of

$$u = \mu[B_k u], \quad \mu \in (0, 1) \quad (11)$$

are *a priori* bounded in E independently of $\mu \in (0, 1)$. Then, by an application of Schaeffer's Theorem, we could conclude the existence of at least a fixed point $u \in E$ for B_k and, in consequence, a solution $u \in \mathcal{E}$ of (1).

If the solutions of (11) are not bounded, then there exist sequences

$$\{u_n\}_{n=1}^\infty, \quad u_n \in E, \quad \|u_n\| \geq n; \quad \{\mu_n\}_{n=1}^\infty, \quad \mu_n \in (0, 1)$$

with

$$\begin{aligned} u'_n(t) + ku_n(t) &= \mu_n [ku_n(t) - F(t, u_n(t))], & \text{a.e. } t \in J', \\ u_n(t_j^+) &= u_n(t_j^-) + \mu_n I_j(u_n(t_j^-)), & j = 1, \dots, p, \\ u_n(0) &= u_n(T). \end{aligned} \quad (12)$$

Now, let $v_n = u_n/\|u_n\|$ so that $\|v_n\| = 1$. Define the functions $\varphi_n(t) = k$ if $|u_n(t)| < r$, and $\varphi_n(t) = F(t, u)/u$ if $|u_n(t)| \geq r$. Thus, $\varphi_n(t) \geq k$ for a.e. $t \in J$, and v_n satisfies

$$\begin{aligned} v'_n(t) + [k(1 - \mu_n) + \mu_n \varphi_n(t)]v_n(t) &= \frac{\mu_n}{\|u_n\|} [\varphi_n(t)u_n(t) - F(t, u_n(t))], & \text{a.e. } t \in J', \\ v_n(t_j^+) &= v_n(t_j^-) + \mu_n \frac{I_j(u_n(t_j^-))}{\|u_n\|}, & j = 1, \dots, p, \\ v_n(0) &= v_n(T). \end{aligned} \quad (13)$$

Hence,

$$\begin{aligned} v'_n(t) + \alpha_n(t)v_n(t) &= \sigma_n(t), & \text{a.e. } t \in J', \\ v_n(t_j^+) &= v_n(t_j^-) + \theta_{n,j}, & j = 1, \dots, p, \\ v_n(0) &= v_n(T), \end{aligned} \quad (14)$$

with $\alpha_n(t) = k(1 - \mu_n) + \mu_n \varphi_n(t)$, $\sigma_n(t) = (\mu_n/\|u_n\|)[\varphi_n(t)u_n(t) - F(t, u_n(t))]$, and $\theta_{n,j} = \mu_n(I_j(u_n(t_j^-))/\|u_n\|)$.

Using the integral representation of the solutions of (5), we have that

$$v_n(t) = \int_0^T g_{\alpha_n}(t, s)\sigma_n(s)ds + \sum_{j=1}^p g_{\alpha_n}(t, t_j)\theta_{n,j}. \quad (15)$$

Now, $\alpha_n(t) \geq k(1 - \mu_n) + \mu_n k = k > 0$ a.e. $t \in J$. In consequence, we get that $0 < g_{\alpha_n}(t, s) \leq g_k(t, s)$ for every $(t, s) \in J \times J$.

On the other hand,

$$|\varphi_n(t)u_n(t) - F(t, u_n(t))| \leq kr + h_r(t), \quad \text{a.e. } t \in J,$$

and

$$|\sigma_n(t)| \leq \frac{kr + h_r(t)}{\|u_n\|} \leq \frac{kr + h_r(t)}{n}.$$

This last inequality implies that the sequence $\{\sigma_n\}_{n=1}^\infty \rightarrow 0$ in $L^1(J)$. Consequently, for every $t \in J$, we have that

$$\left| \int_0^T g_{\alpha_n}(t, s)\sigma_n(s)ds \right| \leq \int_0^T g_k(t, s)|\sigma_n(s)|ds \rightarrow 0, \quad (\text{as } n \rightarrow \infty).$$

Now, condition (10) implies that for every $j = 1, \dots, p$,

$$\lim_{n \rightarrow \infty} \theta_{n,j} = 0,$$

and hence, for every $t \in J$ we obtain the following estimation:

$$\left| \sum_{j=1}^p g_{\alpha_n}(t, t_j)\theta_{n,j} \right| \leq \sum_{j=1}^p g_k(t, t_j)|\theta_{n,j}| \leq \frac{1}{1 - e^{-kT}} \sum_{j=1}^p |\theta_{n,j}| \rightarrow 0, \quad (\text{as } n \rightarrow \infty).$$

From (15), we deduce that $\{v_n\}_{n=1}^\infty \rightarrow 0$ which is a contradiction to the fact that $\|v_n\| = 1$. This concludes the proof.

Now, if $F(t, u) = \lambda u - f(t, u)$, $\lambda > 0$ and f continuous and bounded, then (9) holds for r sufficiently large and $0 < k < \lambda$. If the impulses I_j are continuous and bounded, then (10) is also valid. Hence, our theorem is a generalization of Theorem 3.1 of [9].

Also, if the nonlinearity f and the impulses I_j have sublinear growth, then conditions (9) and (10) are satisfied and we improve Theorem 3.3 in [9].

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